

# Spin Foam Quantization and Anomalies

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**Abstract.** The most common spin foam models of gravity are widely believed to be discrete path integral quantizations of the Plebanski action. However, their derivation in present formulations is incomplete and lower dimensional simplex amplitudes are left open to choice. Since their large-spin behavior determines the convergence properties of the state-sum, this gap has to be closed before any reliable conclusion about finiteness can be reached. It is shown that these amplitudes are directly related to the path integral measure and can in principle be derived from it, requiring detailed knowledge of the constraint algebra and gauge fixing. In a related manner, minimal requirements of background independence provide non trivial restrictions on the form of an anomaly free measure. Many models in the literature do not satisfy these requirements. A simple model satisfying the above consistency requirements is presented which can be thought of as a spin foam quantization of the Husain–Kuchař model.

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## 1. Introduction

In recent years, spin foam models have been established as possible candidates for a quantum theory of gravity (for recent reviews see [1, 2]). They are commonly viewed as covariant (path integral) versions of a canonical quantization and in fact share some features of quantum geometry (in the case of 2+1 a precise relation can be established [3]). As a discretized path integral they can be derived from Plebanski’s action [4] which is a formulation of general relativity as a constrained  $BF$ -theory [5]. Being path integrals of a gauge theory, they have to deal with the anomaly issue: the path integral measure has to be invariant under transformations generated by the constraints. Sometimes it is claimed that a covariant quantization avoids the issue of anomalies which plagued canonical approaches for a long time (but see [6]). However, it is well known that there is also an anomaly problem in path integral quantizations which in spin foam quantizations has just been ignored in most of the existing literature (see, however, [7] for a recent

paper which discusses this issue independently in the example of 3-dimensional  $BF$ -theory). A complete analysis would require an understanding of the continuum limit which has not yet been developed sufficiently. Still, we will see that it is possible to shed light on the problem and to derive conditions for the amplitudes involved in the definition of a spin foam model. To that end we look at the problem from two perspectives. First, we view the discretized version, which is obtained by fixing a space-time triangulation, as a regularization of the path integral, in particular its measure, which would result in the continuum limit. Second, we analyze the restrictions on the spin foam measure imposed by background independence directly at the discrete level.

To explain the part of the anomaly problem studied here we first recall the situation of standard path integrals: There is a prescription which results in a unique measure (up to a constant factor) which is at least formally invariant. This is usually the obvious measure which integrates over all canonical coordinates with constant weight function. After removing the regulator the measure might not be invariant giving rise to quantum anomalies. In any case, it is necessary to use the invariant measure for the regularized version; otherwise the gauge symmetries are broken explicitly and the results are unphysical. Since the formally invariant measure is obvious in most cases, the standard term ‘anomaly’ only refers to the second issue, namely whether or not the measure will remain invariant after removing the regulator. †

In the case of spin foam models, the situation is more involved. First, the constraint algebra is mixed and not closed which will be seen to lead to an additional function in the measure which has not been taken into account previously. Secondly, the space-time discretization obscures the role of the measure and the meaning of invariance in this context. Therefore, even analyzing the formal invariance of a measure requires new techniques which will be provided in the present paper. The usual anomaly problem, which analyzes the invariance after the regulator is removed, will not be touched here since the continuum limit is not understood. We will, however, see that already a formally invariant measure, which is a necessary prerequisite for an anomaly-free continuum measure, puts strong restrictions on amplitudes in the spin foam model. Our definition of a formally invariant *discrete* measure, i.e., amplitudes associated with lower dimensional simplices, is that it must descend from the formally invariant continuum measure along the lines of the spin foam discretization. This does not imply that the continuum formulation is seen as fundamental such that the discretization is introduced only for, e.g., computational purposes. Rather, we view the discrete theory as basic and use the continuum formulation as a tool to derive conditions since it must at least be approximated in certain, semiclassical regimes. This viewpoint concerning the lower

† We warn the reader that anomalies in quantum field theories can even refer to quantities having observable consequences, which has no relation to the discussion here. We are only concerned with anomalies of gauge theories which must be absent for physical viability.

dimensional amplitudes and the continuum measure is the same one as usually assumed for the relation between the vertex amplitude and the continuum action. Provided the required calculations are feasible, this will also fix the formally invariant measure uniquely up to a constant factor, which translates in conditions for the spin foam amplitudes.

An immediate question in the context of anomalies is whether or not a spin foam state sum can be finite.<sup>†</sup> The aim of the present paper is to devise methods for checking a spin foam quantization for formal anomalies, thus obtaining conditions for the lower dimensional amplitudes. While there is general agreement on the vertex (4-simplex) amplitude, which can be viewed as representing the exponentiated action, there are no clear-cut arguments as to which lower dimensional simplex amplitudes should be used<sup>‡</sup>; in the literature, it is largely regarded as being open to choice, maybe constrained by semi-classical issues. This problem is particularly pressing because the question of whether or not a model is finite hinges on the asymptotic behavior of these amplitudes.

Note that our criterion is formulated from the perspective that the fundamental theory is intrinsically discrete. No matter how the approach to a continuum description is performed –via a limit or as a coarse-grained approximation– gauge degrees of freedom have to be removed which is only possible with an invariant measure. It is sometimes argued that finiteness arises because ultraviolet or infrared divergences are regularized by quantum gravitational effects like a minimal length scale (this does in fact occur in canonical quantizations [21]). From our point of view, however, this is not tenable since the anomaly issue is completely unrelated to ultraviolet or infrared divergences.

As a consequence of an invariant measure, a standard path integral quantization cannot lead to finite results without gauge fixing if gauge orbits do not have finite volume (which is to be expected for gravity; to avoid confusion we emphasize here that we are mainly concerned with the diffeomorphism constraints, not with an  $SO(4)$  or  $SL(2, \mathbb{C})$  Gauss constraint).<sup>§</sup> A simple model where this can be illustrated is  $2 + 1$  dimensional  $BF$ -theory. This theory is equivalent to  $2 + 1$  dimensional gravity for non-

<sup>†</sup> A partition function does not need to be finite since it usually drops out of observable expressions due to normalization. Nevertheless, in the context of gravity finite state sum models have been discussed, for which both the partition function and transition amplitudes are finite. In gravity, the viewpoint is different from common field theories since state sums are used directly to compute projection operators to the physical Hilbert space, and not only to compute observables.

<sup>‡</sup> In the case of the Barrett–Crane model the normalization that yields finite amplitudes is naturally selected in the context of the group field theory (GFT) formulation [19, 20]; however, no clear connection with the formal path integral has not been found yet.

<sup>§</sup> Sometimes it is argued that the breaking of active space-time diffeomorphism invariance by a triangulation allows finite results. However, in this argument one has to invoke the usual equivalence of active and passive diffeomorphisms, but the very triangulation which breaks active diffeomorphisms also breaks the correspondence between active and passive transformations.

degenerate triads and thus has the same gauge orbit structure at least on the constraint surface. It turns out, and is commonly accepted, that the spin foam amplitude for  $2 + 1$  dimensional  $BF$ -theory is infinite in accordance with the expectation from path integrals. The discrete symmetries of the simplicial action can be explicitly analyzed [7] and directly linked with the triangulation independence of the spin foam model. An interesting case is  $2 + 1$  gravity with cosmological constant  $\Lambda$ . In this case the action can be written as that of a  $Spin(4)$  Chern-Simons theory whose level  $k$  is given by  $k = 4\pi/\sqrt{\Lambda}$  (see [13] and references therein). A path integral quantization of this theory leads to the Turaev-Viro model defined in terms of a quantum group  $SU_q(2)$  for  $q$  a root of unity related to the cosmological constant by  $q = \exp[2\pi i/(k + 2)]$ . Transition amplitudes turn out to be finite. Although this is often interpreted as a consequence of the infrared cut-off introduced by the quantum deformation, from our viewpoint this is a consequence of the compactness of the gauge group  $Spin(4)$ .

It is not clear however how to generalize this intuition to four dimensions. The gauge properties of  $BF$ -theory in four dimensions are very similar to its 3-dimensional relative. In particular, divergences in the path integral [14] can also be traced back to infinite volume factors coming from the topological gauge symmetry. If we concentrate on the spin foam models for four dimensional gravity that are obtained from an implementation of constraints on the  $BF$  amplitudes (such as Reisenberger[15] or the Barrett–Crane models [16, 17]) the topological gauge symmetry is manifestly broken by the implementation of the constraints. As a result, it was debated whether the remnant gauge symmetries would produce diverging spin foam amplitudes or rather contain “finite volume” gauge orbits. Here we show that minimal requirements of background independence imply the existence of divergences and rule out the finite normalizations proposed in the literature [10, 11, 12, 18].

In the following section, we will introduce a finite dimensional toy model which illustrates the steps of a spin foam quantization mimicking  $BF$ -theory with additional constraints. In Section 3 we discuss the definition of the (formal) path integral for constrained systems. In Section 4 we revisit the spin foam quantization of  $BF$ -theory in three dimensions to introduce notation and review the gauge analysis of the discrete theory performed in [7]. In Section 5 we discuss the definition of the correct path integral measure for 4-dimensional spin foam models defined as constrained BF state sums. We look at the problem from the passive and active diffeomorphism perspectives. In the first case, we reformulate  $BF$ -theory in a way which makes the relation between the path integral measure and the face amplitude obvious. This provides us with a recipe for computing the large spin behavior of amplitudes in spin foams for gravity discussed in Subsection 5.1.2. In Subsection 5.2.1 we discuss the restrictions imposed on the form of the measure by background independence in the active picture. In the case of the Barrett–Crane model, we show that various normalizations proposed in the literature

do not satisfy these requirements and should be regarded as anomalous. This includes the finite normalization introduced in [20]. In Subsection 5.2.2 we define a simple model satisfying those background independence requirements. The latter can be thought of as the spin foam quantization of the Husain–Kuchař model.

## 2. A toy model

To illustrate the importance of choosing the correct measure in spin foam models we first discuss a simple toy model with a finite number of degrees of freedom. It incorporates the essential steps of a spin foam quantization of Plebanski’s action for gravity, which are a field discretization and the solution of a constraint for Lagrange multipliers. Being a system with a finite number of degrees of freedom, the continuum limit cannot be modeled. However, as discussed before, the anomaly issue already requires the correct treatment of the regularization before the continuum limit is taken, which will be illustrated here.

### 2.1. Definition and evaluation

The action of the model is given by

$$S = \int dt(\dot{q}_1 p_1 + \dot{q}_2 p_2 + \lambda_1 q_1 + \lambda_2 q_2 + \xi(\lambda_1 - \lambda_2)) \quad (1)$$

which has two constrained degrees of freedom<sup>†</sup>  $(q_1, q_2)$ , which we assume to live on a circle, with conjugate momenta  $(p_1, p_2)$  and three Lagrange multipliers  $\lambda_1$ ,  $\lambda_2$  and  $\xi$ . Compared with  $BF$ -theory,  $(p_1, p_2, \lambda_1, \lambda_2)$  represents the components of the field  $B$  which contains both physical degrees of freedom and Lagrange multipliers. If we set  $\xi = 0$  resulting in the action

$$S|_{\xi=0} = \int dt(\dot{q}_1 p_1 + \dot{q}_2 p_2 + \lambda_1 q_1 + \lambda_2 q_2),$$

the theory is constrained completely, i.e., both  $q_1$  and  $q_2$  must be zero. There are no degrees of freedom in this case. With unrestricted  $\xi$ , however, the two original Lagrange multipliers are constrained which restores one degree of freedom:  $\lambda_1$  has to equal  $\lambda_2$  and thus only  $q_1 + q_2$  has to be zero whereas the difference is free, which can easily be seen by solving the  $\xi$ -constraint explicitly:

$$S = \int dt(\dot{q}_1 p_1 + \dot{q}_2 p_2 + \lambda_1(q_1 + q_2)).$$

<sup>†</sup> The system is not constrained completely since after solving the constraint obtained after varying  $\xi$  there is only one constraint for two degrees of freedom. The remaining constraint requires  $q_1 + q_2$  to be zero, but otherwise  $q_1 = -q_2$  is free. Since the usual kinetic term is missing, the dynamics then requires  $q_1$  to be constant, which we will also see later in the path integral quantization.

This feature mimics the transition from  $BF$ -theory to gravity where also additional constraints (the simplicity constraints) reduce the freedom of original Lagrange multipliers of  $BF$ -theory and thereby introduce local degrees of freedom.

A spin foam quantization proceeds by quantizing the simple theory whose discretized state sum can be computed explicitly and incorporating the additional constraints at the state sum level. The simple theory (the analog of  $BF$ -theory) here is  $S|_{\xi=0}$  with path integral

$$Z_0 = \int \mathcal{D}^2 q \mathcal{D}^2 p \mathcal{D}^2 \lambda e^{iS|_{\xi=0}} = \int \mathcal{D}^2 q \mathcal{D}^2 p \exp(i \int dt (\dot{q}_1 p_1 + \dot{q}_2 p_2)) \delta(q_1) \delta(q_2) = \int \mathcal{D}^2 p \quad (2)$$

with  $\mathcal{D}^2 q := \mathcal{D} q_1 \mathcal{D} q_2$ . The result is certainly infinite since we are dealing with an unfixed gauge theory. In this case a gauge fixing is simple, but we do not do this because we want to understand the role of a field discretization and the multiplier constraints in this respect.

The path integral for  $S$ , the analog of gravity, in this case can also be obtained explicitly:

$$\begin{aligned} Z &= \int \mathcal{D}^2 q \mathcal{D}^2 p \mathcal{D}^2 \lambda \mathcal{D} \xi e^{iS} = \int \mathcal{D}^2 q \mathcal{D}^2 p \mathcal{D} \lambda_1 \exp(i \int dt (\dot{q}_1 p_1 + \dot{q}_2 p_2 + \lambda_1 (q_1 + q_2))) \quad (3) \\ &= \int \mathcal{D}^2 q \mathcal{D}^2 p \exp(i \int dt (\dot{q}_1 p_1 + \dot{q}_2 p_2)) \delta(q_1 + q_2) = \int \mathcal{D} q_1 \mathcal{D} \Delta p \exp(i \int dt \dot{q}_1 \Delta p) \int \mathcal{D} p' \end{aligned}$$

with  $\Delta p := p_1 - p_2$  and  $p' = p_1 + p_2$ . Computing the remaining integrations we obtain

$$Z = \int \mathcal{D} q_1 \delta(\dot{q}_1) \int \mathcal{D} p' = \delta(q_1^{(0)} - q_1^{(1)}) \int \mathcal{D} p'. \quad (4)$$

Here,  $q_1^{(0)}$  and  $q_1^{(1)}$  represent the initial and the final value of  $q_1$  which are constrained to be equal but free otherwise. Due to the fact that we have only one remaining gauge symmetry after incorporating the  $\xi$ -constraint, we only have one infinite integral left rather than two in (2).

In finite spin foam models one solves the multiplier constraint  $\lambda_1 - \lambda_2 = 0$  at the discretized level and, in some cases, obtains a finite result even *without* fixing the remaining gauge freedom [23]. A spin foam quantization, however, also involves a discretization of space-time which, as already mentioned, is not realized in this finite dimensional model. Still, it is worth checking what effects a field discretization itself can have; effects of the space-time discretization will be discussed later. Therefore, we now discretize  $\lambda_1$  and  $\lambda_2$  which are analogous to  $B$ -field components (we could also discretize the remaining components  $p_1$  and  $p_2$ , without changing our results). In analogy to a spin foam quantization we do this by writing the integral representation of the delta function  $\delta(q_1) = (2\pi)^{-1} \int d\lambda_1 e^{i\lambda_1 q_1}$  as a sum  $\delta(q_1) = (2\pi)^{-1} \sum_{n_1} e^{in_1 q_1}$ . The path integral (2) (absorbing constant factors in the measure) then becomes

$$Z_0 = \int \mathcal{D}^2 q \mathcal{D}^2 p \sum_{\{n_1\}, \{n_2\}} \exp(i \int dt (\dot{q}_1 p_1 + \dot{q}_2 p_2 + n_1 q_1 + n_2 q_2)) \quad (5)$$

where the summation index  $\{n\}$  indicates that  $n$  is not a single number but a function of time, and we are summing over the values at fixed times individually (i.e., this is a discrete analog of the path integral).

At the discrete level the  $\xi$ -constraint implies  $n_1 = n_2$  and we must only sum over those pairs of integers fulfilling this condition in order to obtain a quantization for  $S$  (analogously to summing only over simple representations in a spin foam quantization):

$$Z = \int \mathcal{D}^2 q \mathcal{D}^2 p \sum_{\{n_1\}} \exp(i \int dt (\dot{q}_1 p_1 + \dot{q}_2 p_2 + n_1(q_1 + q_2))) . \quad (6)$$

This is the analog of (3) and its value is, of course, the same as in the calculation with continuous  $\lambda_1$ . In particular, it is infinite. As anticipated, the field discretization and the spin foam like quantization could not take care of the gauge orbit divergence. The divergent integral of (3) has just been replaced by a divergent sum in (6).

## 2.2. Modifying the measure

In spin foam quantizations the issue of convergence hinges on the choice of lower dimensional simplex amplitudes, which can be considered as functions of some components of the discretized  $B$ -field. In our model, however, we do not have any free function available since the path integral result is unique. To include such a function we write our result in the spin foam form (there is still a  $p$ -integral because we chose not to discretize  $p$ )  $Z = \int \mathcal{D}^2 p \sum_{\{n_1\}} V(p_1, p_2, n_1)$  where

$$V(p_1, p_2, n_1) := \int \mathcal{D}^2 q \exp(i \int dt (\dot{q}_1 p_1 + \dot{q}_2 p_2 + n_1(q_1 + q_2)))$$

is the vertex amplitude (analogous to the integration over connections of the discretized  $e^{iS}$ ). A model of lower dimensional amplitudes can now be included by simply inserting a new function  $A(n_1)$  into  $Z$  (more generally,  $A$  could also depend on  $p$ ):

$$Z = \int \mathcal{D}^2 p \sum_{\{n_1\}} A(n_1) V(p_1, p_2, n_1) .$$

Our derivation shows that the face amplitude  $A(n_1)$  is fixed and constant, but let us see what a different function would imply. For illustrative purposes, we choose  $A(n_1) = (2n_1 + 1)^{-2}$  which is finite for all integer  $n_1$ . Now it is easy to see that

$$\begin{aligned} Z' &= \int \mathcal{D}^2 p \sum_{\{n_1\}} (2n_1 + 1)^{-2} V(p_1, p_2, n_1) \\ &= \int \mathcal{D}^2 q \mathcal{D}^2 p \sum_{\{n_1\}} (2n_1 + 1)^{-2} \exp(i \int dt (\dot{q}_1 p_1 + \dot{q}_2 p_2 + n_1(q_1 + q_2))) \\ &= \int \mathcal{D}^2 q \mathcal{D}^2 p \exp(i \int dt (\dot{q}_1 p_1 + \dot{q}_2 p_2 + V_{\text{eff}}(q_1, q_2))) \end{aligned}$$

is finite, where we have the effective potential

$$V_{\text{eff}}(q_1, q_2) = \log(q_1 + q_2 - \pi) - \frac{1}{2}(q_1 + q_2). \quad (7)$$

(We used the Fourier series  $\sum_k (2k+1)^{-2} e^{ik\phi} = -\frac{\pi}{4}(\phi - \pi)e^{-i\phi/2}$  for  $0 \leq x < 2\pi$  and extended with  $2\pi$ -periodicity.) In fact, this is an ordinary path integral for a system of two degrees of freedom in an effective potential  $V_{\text{eff}}$  *without* constraints. We now have to decide if this finite result makes sense and can tell us anything about the original system. The answer is clearly *negative*: The role of the effective potential is completely unclear, and it has nothing to do with the original system. Originally,  $q_1$  and  $-q_2$  have to equal each other but are free otherwise, whereas in the modified system they are independent but subject to motion in a potential. Furthermore, the kind of modification, e.g. the form of the potential, depends on the face amplitude which has no distinguished form other than  $A(n_1) = 1$  which follows from the invariant measure. In conclusion, a finite path integral for an unfixed system with constraints cannot be trusted. (It cannot even be regarded as an approximation since the measure is not just a smeared version of a  $\delta$ -function with support on the constraint surface, but this does not happen at the constraint surface  $q_1 + q_2 = 0$ , but at  $q_1 + q_2 = \pi$ .) In fact, introducing a non-constant amplitude is nothing but introducing an arbitrary function  $A(\lambda_1)$  into the path integral which *breaks the invariance of the measure*. (Note that  $\lambda_1$  serves as a Lagrange multiplier and thus its conjugate momentum  $p_{\lambda_1}$  is implicitly constrained to be zero. The gauge freedom generated by this constraint is broken by introducing an arbitrary function of  $\lambda_1$  into the measure. Consequently, the multiplier  $\lambda_1$  is no longer completely free which also affects the remaining gauge freedom.) This explains why we get a finite result with independent  $q_1, q_2$ ; and it also demonstrates that here a finite model is *anomalous*. As discussed in the Introduction, the space-time discretization, which is not modeled here, presents a possible rescue for finite spin foam models. To check this, we need more general methods which will be introduced in what follows.

### 3. General discussion

Since our model incorporated some of the essential steps of a spin foam quantization of gravity, it suggests that the same conclusions regarding the choice of amplitudes hold true in this more complicated case. In this section we discuss the continuous path integral and the correct measure in the presence of second class constraints, which will be necessary to derive the anomaly-free amplitudes.

The characteristic feature of the gravitational action which is commonly used for a spin foam quantization is the presence of a constraint which restricts the allowed values of Lagrange multipliers appearing in a simpler action. To find the correct measure it



is not sufficient to work solely in a Lagrangian formulation; in particular it is essential to understand the structure of the constraint algebra which can only be achieved in a Hamiltonian analysis. The constraint algebra in this context is always mixed (i.e. neither purely first class nor purely second class) and rather complicated. There are always the usual diffeomorphism constraints of gravity which must form a suitable first class sub-algebra, but in this particular formulation there is also a second class contribution: a constraint which restricts the multipliers of other constraints must be second class. Despite first appearance, even in the toy model the additional constraint is second class. Although the constraints  $C_1 = q_1$ ,  $C_2 = q_2$  and  $C_3 = \lambda_1 - \lambda_2$  Poisson commute, one has to take into account that in this form they are constraints on a *non*-symplectic Poisson manifold with coordinates  $(q_1, p_1; q_2, p_2; \lambda_1, \lambda_2)$  where the standard definitions of Dirac's classification do not apply (see [24] for a discussion and generalized definitions). One can easily introduce an equivalent constrained system which has constraints on a symplectic manifold by adding the momenta  $\pi_1$  and  $\pi_2$  conjugate to the restricted multipliers  $\lambda_1$  and  $\lambda_2$ , together with the constraints  $C_4 = \pi_1$ ,  $C_5 = \pi_2$ . The constraints  $C_I$ ,  $I = 1, \dots, 5$  are then defined on a symplectic manifold and now it is obvious that  $C_3$  does not commute with all constraints. In fact  $C_3$  and  $C_4 - C_5$  form a second class sub-algebra,  $\{C_3, C_4 - C_5\} = 2$ , whereas  $C_1$ ,  $C_2$  and  $C_4 + C_5$  are first class.

In the absence of constraints the invariant path integral measure is given by the determinant of the symplectic form which leads to  $\mathcal{D}q\mathcal{D}p$  for canonical coordinates  $(q, p)$ . A similar treatment is not possible for the multiplier integration since multipliers form a Lagrangian sub-manifold of the extended phase space such that the determinant of their symplectic structure would be zero. The measure is well-defined after solving the second class constraints (and turning the first class constraints into second class ones by fixing the gauge), which leads to the symplectic structure following from the Dirac bracket. For completeness, we will next show how to derive the correct treatment of the multiplier measure by requiring that after solving the constraints in the integral we obtain the determinant of the Dirac symplectic structure [25].

We start with a system with  $2n$  coordinates  $x_i$ ,  $i = 1, \dots, 2n$  on a symplectic phase space  $(M, \omega)$  and  $m$  second class constraints  $C_I$ ,  $I = 1, \dots, m$ . (There might be additional, first class constraints which are not relevant for this section. They can either be gauge fixed and included in the constraints  $C_I$  or be left for later treatment, e.g. factoring out the volume of their gauge orbits. The second possibility is particularly interesting here since a gauge fixing is sometimes claimed to be unnecessary for spin foam models of gravity.) The path integral (where the constraint part has been split off the action  $S = S_0 + \int \sum_I \lambda^I C_I$ ) then is

$$Z = \int \mathcal{D}^{2n}x \sqrt{\det \omega} \mathcal{D}^m \lambda \mu(x) \exp \left( i \int \sum_I \lambda^I C_I \right) \exp(iS_0)$$

with a function  $\mu(x)$  for the multiplier measure which will be determined shortly. This

function must not depend on the  $\lambda^I$  because otherwise the multiplier integration would not yield  $\delta$ -functions of the constraints. With a  $\lambda$ -independent  $\mu$  we can perform the  $\lambda$  integrations explicitly and obtain

$$Z = \int \mathcal{D}^{2n}x \sqrt{\det \omega} \mu(x) \prod_I \delta(C_I) \exp(iS_0).$$

To proceed further we transform from the coordinates  $x_i$ ,  $i = 1, \dots, 2n$  to coordinates  $(y_\alpha; C_I)$  with  $\alpha = 1, \dots, 2n-m$ ,  $I = 1, \dots, m$  (assuming that the constraints are regular and irreducible such that they can be used as local coordinates on  $M$ ). To find the Jacobian of this transformation we use the fact that locally the symplectic manifold  $(M, \omega)$  can be represented as  $(M, \omega) \cong (R, \omega_D) \times_R (P, \Pi_P^{-1})$  using the following notation. The symplectic manifold  $(R, \omega_D)$  is the constraint surface  $R \subset M$  defined by  $C_I = 0$ ,  $I = 1, \dots, m$ , endowed with the Dirac symplectic structure  $\omega_D$ . The manifold  $P$  is given by the image of a neighborhood of a point in  $R$  under the functions  $C_I: M \rightarrow \mathbb{R}$  (i.e.  $P$  is a neighborhood of 0 in  $\mathbb{R}^m$ ; for our purposes it is sufficient to know  $P$  only locally) and coordinatized by  $(C_I)$ ,  $I = 1, \dots, m$ . If the constraint algebra is closed,<sup>†</sup>  $P$  can be defined globally and is a Poisson manifold with Poisson tensor  $\Pi_P$  defined by  $\Pi_P(dC_I, dC_J) := \{C_I, C_J\}$  where the bracket on the right hand side is computed using the symplectic structure  $\omega$  on  $M$  [26]. For second class constraints the inverse of  $\Pi_P$  exists and  $(P, \Pi_P^{-1})$  is a symplectic manifold. If the constraint algebra is not closed, the Poisson tensor  $\Pi_P$  depends not only on the coordinates  $C_I$  of  $P$ , but also on the coordinates of  $R$  such that  $(P, \Pi_P)$  as a symplectic manifold depends on the point in  $R$  chosen for its definition. The right component of the product decomposition of  $M$  then depends on a point in the left component, which is indicated by the subscript  $R$  of the symbol  $\times$ . That the decomposition is valid locally can be shown using the methods of [24] where it has been proven for a closed algebra.

Here we use this local decomposition to factor the original symplectic structure  $\omega = \omega_D \otimes \Pi_P^{-1}$  which allows us to perform the coordinate transformation in the last integral,

$$\begin{aligned} Z &= \int \mathcal{D}^{2n-m}y \mathcal{D}^m C \sqrt{\det \omega_D / \det(\{C_I, C_J\})} \mu(y, C) \prod_I \delta(C_I) \exp(iS_0) \\ &= \int \mathcal{D}^{2n-m}y \sqrt{\det \omega_D} \mu(y, 0) / \sqrt{\det(\{C_I, C_J\})} \exp(iS_0). \end{aligned}$$

Since  $y_\alpha$  are coordinates on the constraint surface, their path integral measure must be given by the Dirac symplectic structure. If there is any other non-constant function besides the exponential of the action, the measure will not be invariant and the path integral will be anomalous. Therefore, the function  $\mu(x)$  has to be  $\sqrt{\det(\{C_I, C_J\})}$

<sup>†</sup> i.e., the right hand side consists of functions of the constraints and thus is constant on the constraint surface

which fixes the free function in the original integral (strictly speaking,  $\mu$  is only fixed on the constraint surface). In the presence of second class constraints, therefore, the path integral to start with is

$$Z = \int \mathcal{D}^{2n}x \mathcal{D}^m \lambda \sqrt{\det \omega} \sqrt{\det(\{C_I, C_J\})} \exp\left(i \int \sum_I \lambda^I C_I\right) \exp(iS_0) \quad (8)$$

which requires a detailed knowledge of the constraint algebra. Note that the determinant of the constraint brackets also appears in this form when first class constraints  $D_\alpha$  are gauge fixed a la Faddeev–Popov, where the other half of the second class constraints  $C_I$  are gauge fixing conditions  $f_\beta$ :<sup>†</sup> In this case the measure contains a function  $\det(\partial f_\alpha / \partial \delta_\beta|_{\delta=0})$  where  $f_\alpha$  is a gauge fixing condition for the first class constraint  $D_\alpha$  and  $\delta_\alpha$  its gauge parameter. Thus,  $\partial f_\alpha / \partial \delta_\beta|_{\delta=0} = \{C_{m/2+\alpha}, C_\beta\}|_{C=0}$  (we assume that the second class constraints are arranged in such a way that the first  $m/2$  are the original first class constraints,  $C_I = D_I$  for  $1 \leq I \leq m/2$ , and the rest are the gauge fixing conditions  $C_I = f_I$  for  $m/2 < I \leq m$ ) and

$$\det(\partial f_I / \partial \delta_J|_{\delta=0}) = \det\{C_I, C_J\}|_{C=0; 1 \leq I \leq m/2; m/2 < J \leq m} = \sqrt{\det\{C_I, C_J\}|_{C=0}}. \quad (9)$$

In existing spin foam quantizations the correct factor for the multiplier integration has not been taken care of; instead any multiplier has been associated simply with a measure  $\mathcal{D}\lambda$  without justification. Note that the additional factor can be ignored when it is constant on the constraint surface, which is always the case for a closed constraint algebra. In particular, our toy model has a closed algebra and so our treatment was correct even though we ignored the additional factor in the measure. For more complicated systems including gravity in the Plebanski formulation, however, this is not expected to be the case.

#### 4. *BF*-theory

*BF*-theory is important in what follows because it appears as an intermediate step in the definition of the gravitational spin foam model. In this section we use it also to illustrate the possible role played by the space-time discretization in the context of anomalies.

The action of *BF*-theory is given by

$$S(B, A) = \int_{\mathcal{M}} \text{Tr}(B \wedge F(A)), \quad (10)$$

where the field  $A$  corresponds to a connection on a principal bundle with compact structure group  $G$  (which will later be taken to be  $SU(2)$ , which gives 3-dimensional

<sup>†</sup> In a covariant formulation one usually chooses a gauge fixing functional depending on all components of the fields, which would require the use of the extended phase space in a canonical picture. For pure first class constraints it is expected that a covariant gauge fixing would be better suited to the spin foam approach; an example can be found in [7] and in Section 4.

Riemannian gravity) and the field  $B$  is a Lie algebra valued 1-form. The local symmetries of the action correspond to the internal gauge transformations

$$\delta B = [B, \omega], \quad \delta A = d_A \omega, \quad (11)$$

for  $\omega$  a Lie algebra valued scalar field where  $d_A$  denotes the covariant exterior derivative, and ‘triad translations’

$$\delta B = d_A \eta, \quad \delta A = 0, \quad (12)$$

where  $\eta$  is a Lie algebra valued function. The first invariance is manifest from the form of the action, while the second is a consequence of the Bianchi identity,  $d_A F = 0$ . If one writes the theory in the Hamiltonian formulation, one observes that the previous symmetries are gauge symmetries in the Dirac sense, i.e., they are generated by the Poisson bracket with the corresponding first class constraints; there are no second class constraints.

Moreover the number of constraints equals the number of configuration variables of the phase space, which implies the theory can only have global degrees of freedom. This can be checked directly by writing down the equations of motion  $F(A) = d_A A = 0$ ,  $d_A B = 0$ . The first equation is solved by flat connections which are locally gauge. The solutions of the second equation are also locally gauge, once the flatness condition ( $F(A) = 0$ ) holds, as any closed form is locally exact.<sup>†</sup>

#### 4.1. Derivation of the spin foam model

To fix our notation, we will now discuss the spin foam quantization of three-dimensional gravity, where  $\mathcal{M}$  is a three-dimensional manifold and  $G = SU(2)$ , and later mention necessary changes for four dimensions. The quantization of  $BF$ -theory is done by replacing the manifold  $\mathcal{M}$  with an arbitrary cellular decomposition  $\Delta$ . We also need the notion of the associated dual 2-complex of  $\Delta$  denoted by  $\mathcal{J}_\Delta$ . The dual 2-complex  $\mathcal{J}_\Delta$  is a combinatorial object defined by a set of vertices  $v \in \mathcal{J}_\Delta$  (dual to 3-cells in  $\Delta$ ) edges  $e \in \mathcal{J}_\Delta$  (dual to 2-cells in  $\Delta$ ) and faces  $f \in \mathcal{J}_\Delta$  (dual to 1-cells in  $\Delta$ ).

The fields  $B$  and  $A$  have support on these discrete structures by representing the  $su(2)$ -valued 1-form  $B$  as an assignment of a  $B \in su(2)$  to each 1-cell in  $\Delta$  the connection  $A$  as an assignment of a group element  $g_e \in SU(2)$  to each edge in  $\mathcal{J}_\Delta$ . The action of the simplicial theory is given by  $S = \sum_{f \in \mathcal{J}_\Delta} B_{\ell_f} U_f$ , where  $B_{\ell_f}$  is the Lie algebra element associated to the 1-simplex  $\ell_f \in \Delta$ , dual to the face  $f \in \mathcal{J}_\Delta$ , and  $U_f = g_e^1 g_e^2 \cdots g_e^N$  is the discrete holonomy around  $f \in \mathcal{J}_\Delta$ . Since  $\ell_f$  is in one-to-one correspondence with  $f \in \mathcal{J}_\Delta$ , from now on we denote  $B_{\ell_f}$  simply as  $B_f$ .

<sup>†</sup> As is well known, one can easily check that the infinitesimal diffeomorphism gauge action  $\delta B = \mathcal{L}_v B$ , and  $\delta A = \mathcal{L}_v A$ , where  $\mathcal{L}_v$  is the Lie derivative in the  $v$  direction, is a combination of (11) and (12) for  $\omega = v^a A_a$  and  $\eta_b = v^a B_{ab}$ , respectively, acting on the space of solutions.

The partition function is defined as

$$\mathcal{Z}(\Delta) = \int \prod_{f \in \mathcal{J}_\Delta} dB_f \prod_{e \in \mathcal{J}_\Delta} dg_e \exp(i \operatorname{Tr} [B_f U_f]), \quad (13)$$

where now  $dB_f$  is the regular Lebesgue measure on  $su(2) \cong \mathbb{R}^3$ ,  $dg_e$  corresponds to the invariant measure on  $SU(2)$ .

Integrating over  $B_f$ , we obtain

$$\mathcal{Z}(\Delta) = \int \prod_{e \in \mathcal{J}_\Delta} dg_e \prod_{f \in \mathcal{J}_\Delta} \delta(g_e^1 \dots g_e^N), \quad (14)$$

where  $\delta$  corresponds to the delta distribution defined on  $\mathcal{L}^2(SU(2))$ .

The integration over the discrete connection ( $\prod_e dg_e$ ) can be performed if one expands the delta function in the previous equation using harmonic analysis on the group. Using the Peter–Weyl decomposition, the  $\delta$ -distribution becomes

$$\delta(g) = \sum_{j \in \operatorname{irrep}(SU(2))} \Delta_j \operatorname{Tr} [\rho_j(g)], \quad (15)$$

where  $\Delta_j$  denotes the dimension of the unitary representation  $j$ , and  $\rho_j(g)$  is the corresponding representation matrix. The partition function (14) becomes

$$\mathcal{Z}(\Delta) = \sum_{\mathcal{C}_f: \{f\} \rightarrow \{j\}} \int \prod_{e \in \mathcal{J}_\Delta} dg_e \prod_{f \in \mathcal{J}_\Delta} \Delta_{j_f} \operatorname{Tr} [\rho_{j_f}(g_e^1 \dots g_e^N)], \quad (16)$$

where  $\mathcal{C}_f: \{f\} \rightarrow \{j\}$  represents the assignment of irreducible representations to faces in the dual 2-complex  $\mathcal{J}_\Delta$ . Each particular assignment is referred to as a *coloring*. The summation is then over colored 2-complexes (spin foams).

If the  $SU(2)$  group element  $g_e$  corresponds to an  $n$ -valent edge  $e \in \mathcal{J}_\Delta$ , i.e., an edge bounding  $n$  faces, there are  $n$  representation matrices evaluated on  $g_e$  in (14). The relevant integral is

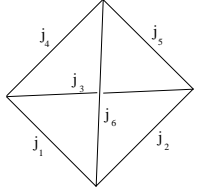
$$\int dg \rho_{j_1}(g) \otimes \rho_{j_2}(g) \otimes \dots \otimes \rho_{j_n}(g) = \sum_{\iota} C_{j_1 j_2 \dots j_n}^{\iota} C_{j_1 j_2 \dots j_n}^{*\iota}, \quad (17)$$

i.e., the projector onto  $\operatorname{Inv}[\rho_{j_1} \otimes \rho_{j_2} \otimes \dots \otimes \rho_{j_n}]$ . On the RHS we have chosen an orthonormal basis of invariant vectors (intertwiners) to express the projector. Integrating over the connection (14) becomes

$$\mathcal{Z}(\Delta) = \sum_{\mathcal{C}_f: \{f\} \rightarrow \{j\}} \sum_{\mathcal{C}_e: \{e\} \rightarrow \{\iota\}} \prod_{f \in \mathcal{J}_\Delta} \Delta_{j_f} \prod_{v \in \mathcal{J}_\Delta} A_v(\iota_v, j_v), \quad (18)$$

where  $A_v(\iota_v, j_v)$  is given by the appropriate trace of the intertwiners  $\iota_v$  corresponding to the edges bounded by the vertex and  $j_v$  are the corresponding representations. This amplitude is given in terms of  $SU(2)$   $3Nj$ -symbols. When  $\Delta$  is a simplicial complex then all the edges in  $\mathcal{J}_\Delta$  are 3-valent and vertices are 4-valent. Consequently, there are 3 representation matrices for all edges in (17) and the corresponding amplitude is given

by the contraction of the corresponding four 3-valent intertwiners, i.e., a  $6j$ -symbol. In that case the partition function takes the familiar Ponzano–Regge form

$$\mathcal{Z}(\Delta) = \sum_{\mathcal{C}_f: \{f\} \rightarrow \{j\}} \prod_{f \in \mathcal{J}_\Delta} \Delta_{j_f} \prod_{v \in \mathcal{V}_\Delta} \text{tetrahedron}(j_1, j_2, j_3, j_4), \quad (19)$$


where the tetrahedron corresponds to the graphical representation of the  $6j$ -symbol.

In the next section we will analyze the case of gravity in four dimensions. The models of interest are defined in terms of constrained 4-dimensional  $BF$ -theory which for Euclidean signature has the gauge group  $SO(4) \cong SU(2) \times SU(2)$ . The discretization of  $BF$ -theory in four dimensions is analogous to that of 3-dimensional  $BF$ -theory. The main difference is that the  $B$  field is now a Lie algebra valued 2-form and so is discretized by the assignments of Lie algebra elements  $\{B_{t_f}\}$  to the 2-dimensional surfaces defined by the triangles  $t_f \in \Delta$ . Triangles are in one-to-one correspondence with faces  $f \in \mathcal{J}_\Delta$ . The connection is discretized in precisely the same way as in three dimensions; namely, by assigning group elements  $g_e$  to edges  $e \in \mathcal{J}_\Delta$ . Upon integration over  $\{B_f\}$  and the  $\{g_e\}$  the amplitudes can be expressed as a spin foam sum similar to (19), where the  $6j$ -symbol is replaced by a  $15j$ -symbol represented by a 4-simplex (see [1] for details and references). Since any  $SO(4)$ -representation can be decomposed into a product of two  $SU(2)$ -representations, the face amplitude is now  $\Delta_{(j_1, j_2)} = (2j_1 + 1)(2j_2 + 1)$ .

#### 4.2. Gauge fixing for 3d gravity

The expression (18) is generically divergent due to the (non-compact) gauge freedom (12) [7]. Some of the  $B$  integrations in (13) – or equivalently some of the  $\delta$ -functions in (14) – are thus redundant. In addition to the standard  $SU(2)$  gauge invariance of the action corresponding to (11), there is a discrete analog of (12) given by

$$\delta B_{\ell_f} = \eta_v - [\Omega_{\ell_f}^v, \eta_v] \text{ if } v \in \ell_f \quad \text{and} \quad \delta B_{\ell_f} = 0 \text{ if } v \notin \ell_f \quad (20)$$

where  $\eta_v$  is a Lie algebra element associated to the vertex  $v \in \Delta$  and  $\Omega_{\ell_f}^v$  is also in the Lie algebra and can be explicitly given in terms of the logarithm of the elements  $\{U_{f'}\}$  for  $f' \neq f$  and contained in the set of faces that form the dual bubble in  $\mathcal{J}_\Delta$  around the vertex  $v$  [7]. The previous transformation is a symmetry of the simplicial action due to the discrete version of the Bianchi identity stating that the (ordered) product of  $U_f$  around a bubble is equal to the identity.

Assuming  $\Delta$  is path connected we can set  $B = 0$  along a contractible (within  $\Delta$ ) path  $L$  containing all vertices in  $\Delta$  using the gauge freedom (20). This fixes this gauge freedom completely and contributes to the measure with the Faddeev–Popov determinant

$$\det(\partial f_I / \partial \delta_J|_{\delta=0}) = (1 + |\Omega_{\ell_f}^v|^2) \quad (21)$$

which only depends on the connection  $\{g_e\}$ . Integration over the  $B$  field produces the curvature delta functions as in (14) which in turn imply  $\Omega_{\ell_f}^v = 0$  and hence trivial Faddeev–Popov factors.

The result is very simple: we have to drop out all the  $\delta$ -functions in (14) corresponding to faces  $f \in \mathcal{J}_\Delta$  dual to 1-simplexes in  $L$  which are redundant leading to divergences in the non gauge fixed formulation. The gauge fixing is then analogous to changing the discretization. In this precise sense we find a connection between discretization independence of the partition function and the gauge freedom (20).

## 5. Spin foam measure

As we have seen in the previous section, the well-known amplitudes of the spin foam model of  $BF$ -theory can be seen as emerging from a concrete transition from the continuum measure of a path integral. In gravity models which are defined as constrained  $BF$ -theories (of the type of Barrett and Crane’s), this direct construction of the measure is not available. This is so because the simplicity constraints are imposed on  $BF$  amplitudes after the integration over the (discrete)  $B$ -field has been performed. At this stage, one is already dealing with a discrete state sum—where  $B$  configurations are replaced by irreducible unitary representations—and the connection with the formal continuous measure is lost.

In this section, we will first present a method to impose the simplicity constraints (or, more generally, second class constraints in a spin foam quantization) in such a way that there is always a clear connection to the continuum measure. It is based on a passive interpretation where the (diffeomorphism) gauge transformations do not change the discretization, but rather values of the fields. Explicit calculations in the case of gravity, however, so far look complicated and we will not pursue a calculation of the amplitudes here [22]. Nevertheless, one can hope that at least their asymptotic behavior for large labels can be found easily, which would allow us to see whether or not the state sum will be finite.

Independently, we can use the active picture where diffeomorphisms change the discretization. There are now moves which lead to a different discretization but a configuration which has to be considered as physically equivalent to the original one. Background independence then requires that the amplitudes do not change under those moves, which as we will see in Subsection 5.2.1 imposes non trivial restrictions on the measure. In Subsection 5.2.2 we present a toy model of a background independent spin foam.

Anomaly freedom and background independence both restrict the lower dimensional amplitudes of a spin foam. Though not necessarily equivalent, both notions are clearly related since a model which has an anomaly for the diffeomorphism gauge group cannot

be background independent, and vice versa. We expect the conditions for the amplitudes to have similar consequences, which can serve as a non-trivial consistency check for the calculations. This will be shown later to be realized in the Husain–Kuchař model.

### 5.1. Passive picture

In the standard derivation of the spin foam models for  $BF$ -theory there is a clear-cut relationship between the path integral measure and spin foam amplitudes coming directly from integrating the  $B$ -field in (13) and using the Peter–Weyl decomposition of the  $\delta$ -distribution. In the next subsection we will re-derive part of the spin foam quantization of  $BF$ -theory with different methods which show how the labels  $j$  arise from a discretization of  $B$ . This will allow us to propose a recipe for the construction of the correct measure in the case of gravity in Subsection 5.1.2.

*5.1.1.  $BF$ -theory in polar coordinates* A main ingredient of the spin foam quantization of  $BF$ -theory in three or four dimensions is the formula

$$(2\pi)^{-1} \int dB \exp(2i \operatorname{tr}(Bg)) = \delta(g) = \sum_j (2j+1) \operatorname{tr} \rho_j(g) \quad (22)$$

where  $g \in SU(2)$ ,  $B \in su(2)$ . The continuous values of  $B$  are effectively replaced by discrete values  $j$ , but the exact correspondence remains unclear. In particular, there are three independent components in  $B$ , but only one discrete label  $j$ . The following calculations will now show that  $j$  is the discretized radial component of  $B$  in polar coordinates whereas the angular components are integrated out.

We write the  $su(2)$  element  $B = rn^i \tau_i$  with  $n = (\sin \vartheta \cos \varphi, \sin \vartheta \sin \varphi, \cos \vartheta)$  in polar coordinates  $(r, \vartheta, \varphi)$  where  $\tau_j = i/2 \sigma_j$  with Pauli matrices  $\sigma_j$ . To simplify the calculation we also choose the  $SU(2)$  element in the gauge  $g = \exp c\tau_3 = \cos \frac{c}{2} + 2\tau_3 \sin \frac{c}{2}$  without loss of generality (thanks to gauge invariance of the trace in (22)). This leads to

$$\delta(g) = (2\pi)^{-1} \int dr d\vartheta d\varphi r^2 \sin \vartheta e^{-2ir \cos \vartheta \sin c/2} = i \int dr \frac{r}{2 \sin \frac{c}{2}} (e^{-2ir \sin c/2} - e^{2ir \sin c/2}) \quad (23)$$

whereas the discrete form is

$$\delta(g) = \sum_j (2j+1) \frac{\sin(j + \frac{1}{2})c}{\sin \frac{c}{2}} = \sum_j \frac{2j+1}{2i \sin \frac{c}{2}} (\exp(i(2j+1)\frac{c}{2}) - \exp(-i(2j+1)\frac{c}{2})).$$

One thus obtains the discrete version of the  $\delta$ -function by integrating out  $\vartheta$  and  $\varphi$ , and replacing the continuous  $r$  with the discrete  $j + \frac{1}{2}$ . (One also has to replace  $\sin \frac{c}{2}$  with  $\frac{c}{2}$ , but since both expressions represent a  $\delta$ -function in  $c$ , they can be regarded as identical.) This clarifies the relation between continuous values for  $B$  and the discrete  $j$ .

The calculation can also be used to show that the standard  $BF$ -face amplitude agrees with the one derived from the invariant path integral measure and thus is anomaly-free.



For Euclidean gravity,  $SO(4)$   $BF$ -theory is used which can be written as a state sum with  $SU(2)$ -valued variables after using  $SO(4) \cong SU(2) \times SU(2)$ . For each copy of  $SU(2)$  on every face with spin  $j_f$  one has a factor contributing to the face amplitude by  $2j_f + 1$  as in the last result in (23) where a single factor  $r$  remains in the measure after integrating over  $\vartheta$ . Discretizing  $r$  then yields the correct face amplitude (Section 4.1).

*5.1.2. A general recipe* For theories more complicated than  $BF$ -theory, but still written as constrained  $BF$ -theory, we have to combine the result of Section 3, which tells us the correct continuum measure in the presence of second class constraints, with a transition from the continuum measure to discrete amplitudes. Since the analysis must be at the canonical level and some field components are distinguished as Lagrange multipliers, in general such a measure will break manifest covariance. We also have to expect additional functions which result from inserting  $\delta$ -functions imposing the constraints into the path integral. The constraint algebra tells us what function we have to include to obtain the correct continuum measure as in Eq. (8), and there can be additional functions coming from Jacobians if the constraints do not directly restrict the integration variables but a more complicated functional of them. The constraint structure of Plebanski's theory has been studied in detail in [22].

A possible strategy is to use the formulation of  $BF$ -theory in polar  $B$ -coordinates where we have seen that the spin foam model with the correct face amplitude arises from integrating out the  $B$ -angles and discretizing the radial  $B$ -coordinate. Additional constraints restricting the  $BF$ -theory then arise from the action via integrating over the corresponding Lagrange multipliers which results in  $\delta$ -functions inserted into the path integral. We have seen that the multiplier integration requires a special measure which can be computed if the constraint algebra is known. Next, we write both this function and the  $\delta$ -functions in the polar  $B$ -coordinates, integrate over the  $B$ -angles and discretize the radial  $B$ -coordinate just as we did in  $BF$ -theory without any additional constraints. In general, this will result in an additional functional besides the vertex amplitude in the spin foam model which depends on the spins  $j$ .

In practice, the calculation, in particular the computation of the constraint algebra and the integration over the angular  $B$ -coordinates, will be complicated (several faces are coupled by the simplicity constraint). To illustrate the type of integrals involved we have included an example of constrained  $BF$ -theory in the appendix. It is also not easy to decide which part of the constraint algebra one has to consider. All constraints taken together form a mixed system containing first class constraints. Usually, as in Section 3, one would have to gauge fix the first class constraints resulting in a pure second class algebra. However, for gravity a gauge fixing is not known, and it is hoped that spin foam models can avoid the need to introduce an explicit gauge fixing by using a space-time discretization. In the following subsection we will see however that simple

considerations of background independence with respect to active diffeomorphisms imply the existence of bubble divergences which are naturally associated to infinite gauge volume contributions. Diffeomorphisms in fact seem not to be completely gauge fixed by the discretization and the situation might be similar to that of  $BF$ -theory reviewed in the previous sections. One then would have to find a second class sub-algebra of the full constraint system plus the appropriate gauge fixing constraint in order to compute the correct measure.

It seems necessary to look for a description of gauge symmetries and constraint algebra that would be directly defined at the discrete level. In the previous section we have reviewed how this can be done in the case of  $BF$ -theory. Indeed, in order to regularize the path integral Freidel and Louapre had to introduce a gauge fixing of the topological symmetry (20) which in turn modifies the measure by the Faddeev-Popov determinant (21). In the case of this topological theory the modification is trivial and the factor is independent of the fields reducing simply to unity. In the case of gravity one should expect a non trivial dependence on spins consistent with a theory with local excitations. It is clear from this example that even when the gauge symmetries of the discrete action retain some similarities with the continuum ones their action can be only interpreted at the discrete level. An equivalent analysis in the case of 4-dimensional Plebanski theory seems necessary in order to implement the general prescription of Subsection 5.1.2 for the construction of the measure and hence settle the issue of lower dimensional simplex amplitudes. In such a context, the first class part of the constraint algebra might not even be first class in the continuum sense as results of Gambini and Pullin show [8, 9]. In this case a direct application of our recipe (or a slight generalization) should be feasible.

## 5.2. *Active picture*

In this section we will show how the requirement of background independence and diffeomorphism invariance from the active point of view imposes restrictions on the spin foam amplitudes with important consequences. For concreteness, we will focus the attention on the Barrett–Crane model for quantum gravity in the formulation of simplicial quantum gravity presented in [4].

*5.2.1. Background independence* The requirements imposed here on the spin foam amplitudes can be viewed as a 4-dimensional generalization of the notion of cylindrical consistency and diffeomorphism invariance in the canonical formulation of loop quantum gravity [27]. According to background independence, the cellular decomposition used to represent the space-time manifold does not carry any physical information. Gravitational degrees of freedom are encoded in the labeling of faces in the dual 2-

complex with irreducible representations of the corresponding internal gauge group: a spin foam. As a consequence there remains some redundant information in a spin foam defined on a particular 2-complex which links the ‘physical’ configuration with the discretization on which it has been defined. The background independent information is encoded in appropriate equivalence classes of spin foams. These equivalence classes have to be introduced if one wants to think of spin foams as morphisms in the spin network category [28], and are defined by the following moves:

- (i) *(Piecewise linear) maps preserving the cell-complex and its coloring*
- (ii) *Subdivision*
- (iii) *Orientation reversal*

A detailed definition can be found in [28]. These moves can be interpreted as the counterpart of diffeomorphisms in simplicial quantum gravity (with perhaps the addition of more equivalence relations if the remnant of diffeomorphism invariance is larger as in  $BF$ -theory). Transformations given by arbitrary combinations of the moves are clearly limited and the resulting equivalence relation is much weaker than complete discretization independence which would be the requirement for a topological quantum field theory. The most important move is the second one since the others leave the number and connections of simplices invariant. An illustration of the limited scope of the moves can be seen in Fig. 1 where only one new edge and two new vertices are created while the rest is unchanged. Thus, the discretizations have to be regarded as defining equivalent backgrounds which are just parameterized differently. (This point is different from lattice gauge theories in a given background where a new edge would probe new degrees of freedom.)

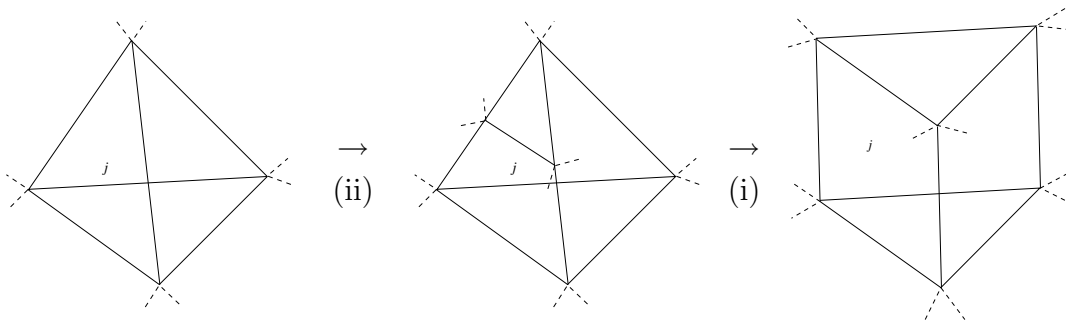
The natural amplitude for faces in spin foam models is given by the Plancherel measure arising in the harmonic analysis on the corresponding internal gauge group. This can be derived in various ways and it is directly linked with the notion of locality of spin foams [15], namely that degrees of freedom communicate along faces by boundary data given by the connection. If the face amplitude  $A_f(j)$  is given by the Plancherel measure (e.g.,  $A_f(j) = (2j + 1)^2$  in the case of the Riemannian Barrett–Crane model) then the arbitrary subdivision of a face  $f \in \mathcal{J}_\Delta$  does not change the amplitude. †

The previous analysis raises the question of whether we can find more stringent conditions on spin foam amplitudes by solely imposing background independence. The simplest consistency condition can be obtained by the requirement that the bubble

† In the canonical picture of loop quantum gravity the boundary data is given by  $SU(2)$  connections for which the Plancherel measure induces a face amplitude  $A_f = 2j + 1$ . In spin foam models based on  $Spin(4)$   $BF$ -theory such a face amplitude could arise as a result of the implementation of second class constraints as discussed in Section 5.1.2.

spin foams of Figures 1 and 3 have the same amplitude. The figure represents spin foams where most of the faces are labeled by the trivial representation except for the shown bubbles, labeled by the simple representation  $\rho = j \otimes j^*$ . These spin foams are clearly equivalent under the above moves and the figures illustrate the sequence of moves that relate them. This is also in agreement with our intuitive notion of background independence (see [1] for more discussion): given that the underlying 2-complex does not carry any geometrical information and that geometry degrees of freedom are represented by the labeling of faces with spins and its intrinsic combinatorics there is no background independent way to distinguish between these bubble spin foam excitations. Their apparent difference is linked to the fiducial background 2-complex used to represent them and should not play any physical role. (This is true only if, as assumed, all outside faces are labeled by the trivial representation; otherwise they will not be related by the moves defined above. Again, this illustrates that the equivalence relation used here is much weaker than that for topological field theories.)

For example, the Barrett–Crane model is a definition of the 4-simplex or vertex amplitude for quantum gravity up to an overall factor. If we normalize the vertex amplitude in some arbitrary way we can shift the ambiguity to the value of the edge amplitude. In addition we assume the edge amplitude to be local in the sense that it only depends on the values of representations that label the corresponding edge. This assumption is in correspondence to the models defined in the literature.<sup>†</sup> The normalization we choose is that for which the Barrett–Crane intertwiner is a norm-one-vector in the Hilbert space where  $\text{Inv}[\rho_1 \otimes \rho_2 \otimes \rho_3 \otimes \rho_4]$  acts. This normalization is naturally obtained in the implementation of Plebanski’s constraints on the  $BF$  partition function [4].



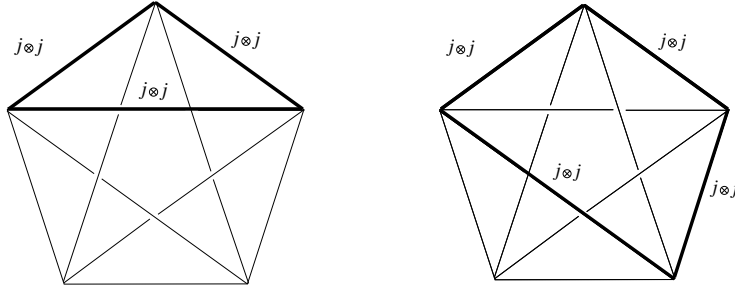
**Figure 1.** Vacuum bubbles to be physically equivalent in an anomaly free spin foam model. Their equivalence constrains the possible behavior of the face and edge amplitudes. We represent the sequence of moves that relate the bubble on the left with that on the right.

<sup>†</sup> However, it is possible that non local modifications of the amplitude could arise as a result of the implementation of constraints as in Section 5.1.2. This possibility will not be explored here.

Let us first study the bubble amplitudes in Figure 1. The first bubble from left to right has four vertices corresponding to the Barrett–Crane  $10j$ -symbol illustrated on the left diagram in Figure 2 containing only three edges labelled by non trivial simple representations  $j \otimes j$ . The value of this vertex can be easily evaluated [1]. It corresponds to the value of the trace of the three non-trivial normalized intertwiners appearing in the triangular loop on the left of Figure 2. If we denote by  $\iota$  the corresponding normalized 2-intertwiner it is obvious that  $\iota = (2j + 1)^{-1} \mathbb{I} \otimes \mathbb{I}$ , where  $\mathbb{I}$  is the identity in the vector space corresponding to the  $SU(2)$  representation  $j$ . Therefore the result is

$$\text{Tr}[\iota \cdot \iota \cdot \iota] = \frac{1}{(2j + 1)^3} \text{Tr}[\mathbb{I} \otimes \mathbb{I}] = \frac{1}{2j + 1}.$$

The amplitude of the tetrahedral bubble is then  $B_1(j) = (2j + 1)^{-4} A_e(j)^6 A_f(j)^4$ , where



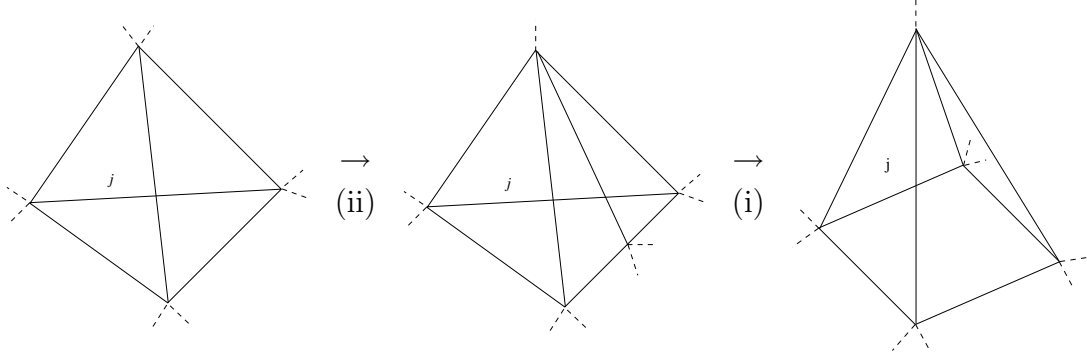
**Figure 2.** Relevant  $15j$ -symbols for  $B_1(j)$ ,  $B_2(j)$  and  $B_3(j)$ . Thin lines are trivial representations.

$A_e(j)$  and  $A_f(j)$  are the so far undetermined edge and face amplitudes. The exponents are: 6 for the six edges and 4 for the four faces of the tetrahedral bubble. We have given already an argument for the value of  $A_f(j)$  which we will re-derive here from the background independence condition. In the case of the bubble diagram on the right of Figure 1 we have six vertices corresponding to the same  $10j$ -symbol as before, so the amplitude for the prism bubble is  $B_2(j) = (2j + 1)^{-6} A_e(j)^9 A_f(j)^5$ .

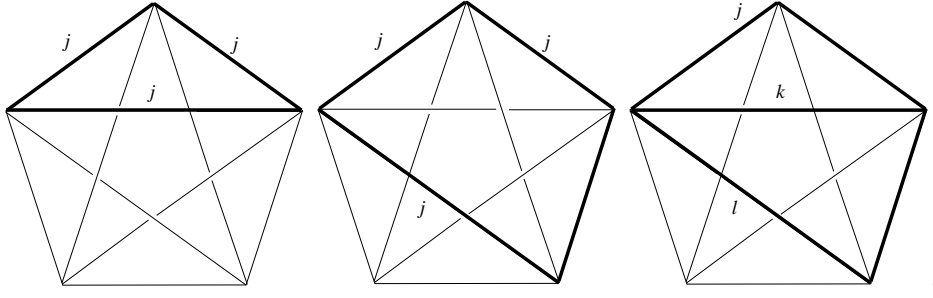
Finally in the case of the bubble spin foam on the right of Figure 3 we have four vertices of the previous type plus the vertex on the top whose  $10j$ -symbol is illustrated in the diagram on the right of Figure 2 which evaluates to  $(2j + 1)^{-2}$ . With this the amplitude of the pyramid bubble is  $B_3(j) = (2j + 1)^{-6} A_e(j)^8 A_f(j)^5$ . The requirement  $B_1(j) = B_2(j) = B_3(j)$  fixes the values of  $A_e(j)$  and  $A_f(j)$  uniquely to

$$A_e(j) = 1 \quad \text{and} \quad A_f(j) = (2j + 1)^2. \quad (24)$$

The previous bubble spin foams are particularly easy to compute and helpful to explain the intuitive idea behind our consistency requirement. There is a more general statement of this property to be satisfied by any background independent



**Figure 3.** Sequence of subdivision and piecewise linear transformation relating the tetrahedral bubble with the pyramidal one.



**Figure 4.** Vertex contributions to the bubble amplitudes above (thin lines represent edges labeled with the trivial representation). From left to right their value is given by  $(2j+1)^{-1}$ ,  $(2j+1)^{-2}$  and  $(2j+1)^{-1}(2l+1)^{-1}$  in the Riemannian Barrett–Crane model if we normalize the corresponding intertwiners.

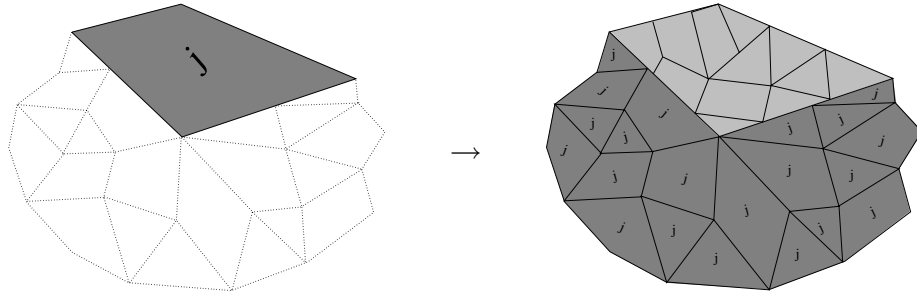
spin foam. Namely, spin foam amplitudes are required to be invariant under the arbitrary subdivision of their faces. Equivalently, if we deform (by a piece-wise linear homeomorphism<sup>†</sup>) a colored face by coloring with the same spin adjacent faces in the 2-complex (previously labeled by the trivial representation) the amplitude should remain invariant, Figure 5. We see that with the normalization found above the Barrett–Crane model satisfies this necessary condition for background independence as the amplitude of the composite face  $A_f^{\text{com}}(j)$  can be easily shown to be given by

$$A_f^{\text{com}}(j) = (2j+1)^{2n_v-2n_e+2n_f} = (2j+1)^{2\chi} = A_f(j), \quad (25)$$

where  $n_e$  and  $n_f$  is the number of internal edges and faces of the composite face. We see that (24) yields an invariant amplitude.

The analysis here fixes the value of the face amplitude to be given by the Plancherel measure of the corresponding gauge group. It is important to notice however that the

<sup>†</sup> For an extensive analysis of role of piece-wise linear homeomorphisms as opposed to diffeomorphism as basic symmetry of quantum gravity see [29].



**Figure 5.** Two equivalent spin foam configurations. The dotted 2-cells on the left are labelled by the trivial representation.

edge amplitude found here is only valid for the degenerate situations in which we have vertex configurations of the form illustrated in Figure 4. These situations are degenerate in the sense that the simplicity constraints of that reduced  $BF$ -theory to gravity are trivial. In the general situation one expects the value of the edge amplitude to differ from the trivial value obtained here. It is precisely here where the appropriate Faddeev–Popov factors advocated in Sections 3 and 5.1.2 will play an important role.

The requirements studied here should be met by any theory admitting a spin foam quantization. In particular it is easy to see that  $BF$ -theory in any dimension would satisfy them. This is however to some extent trivial as  $BF$ -theory is topological and has a finite number of degrees of freedom. The following is a simple example of an (in this sense) background independent spin foam model for a theory with infinitely many degrees of freedom.

*5.2.2. An anomaly free toy model: a trivial example* In this section we define a spin foam model satisfying the above minimal requirements of background independence whose elementary amplitudes are very simple and yet lead to a model that is not topological. The model is tailored to produce a physically interesting model that can be thought of as a spin foam quantization of the Husain–Kuchař model [30].

The action of the Husain–Kuchař model is given by

$$S[e, A] = \int_M e^i \wedge e^j \wedge F^k(A) \epsilon_{ijk}, \quad (26)$$

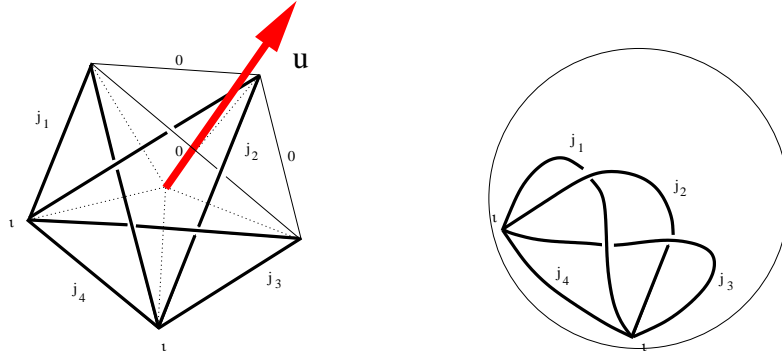
where  $M$  is a 4-dimensional manifold,  $A$  is an  $SU(2)$  connection and  $e^i$  is a dreibein field. As it is shown in [30] one can use the orientation 4-form,  $\epsilon_{abcd}$ , to define a densitized vector field  $u^a$  which is orthogonal to the triad. The previous action can be written as  $SU(2)$ -BF-theory in four dimensions, namely

$$S[e, A] = \int_M B_k \wedge F^k(A), \quad (27)$$

supplemented by the constraint  $B_{ab}^k u^a = 0$  for some given vector field  $u^a$ . In addition to the vector field  $u^a$  we must provide an auxiliary space-time foliation which is

equivalent to say that we are given  $u_a$  such that  $u_a u^a = 1$ . If we define the triad  $e_i^a$  as  $\det(e) e_i^d \equiv B_{ab} u_c \epsilon^{abcd}$ , where  $\det(e)$  is defined using  $\epsilon_{abc} = \epsilon_{abcd} u^d$ , then it is easy to show that  $B_{ab}^i = \epsilon^i_{jk} e_a^j e_b^k$  which when replaced in the BF-theory action reproduces the Husain-Kuchař action.

For a spin foam quantization we can assume that the model is defined on a simplicial decomposition. In this case all vertices in the dual 2-complex are 5-valent and boundary graphs have 4-valent nodes. The standard  $BF$ -theory amplitude is the  $15j$ -symbol constructed with normalized intertwiners (where  $j$  are unitary irreducible representations of the corresponding compact group  $G$ ). Now the constraint  $B_{ab}^k u^a = 0$  is implemented by requiring the vertex amplitude to vanish unless at least three representations labeling the links forming a triangle in the graphical representation of the  $15j$ -symbol are trivial, in which case it is given by the standard  $15j$ -symbol evaluation. The idea is illustrated in Figure 6. Choosing a triangle on a 4-simplex amounts to choosing a direction  $u$  in spacetime. The components of the  $B$ -field on the faces that are bounded by this triangle must vanish according to the constraint  $B_{ab}^k u^a = 0$  which is achieved by setting  $j = 0$  for the corresponding faces. The resulting



**Figure 6.** On the left: geometric interpretation of the constraints. Such spin foam corresponds exactly to an edge carrying an intertwiner  $\iota$  between four representations  $j_1, \dots, j_4$  entering the 4-simplex and leaving it without change. On the right: the continuum representation.

spin-network on the boundary of the 4-simplex is simply a  $\theta$ -spin-network. Therefore the allowed transitions are very rigid which implies rather simple configurations.

This model can produce infinite transition amplitudes whenever a ‘vacuum’ bubble of the kind represented in Figures 1 or 3 is created. The model is background independent in the sense above if we set the face amplitude as the corresponding Plancherel measure of the Lie group of interest to the power of the Euler characteristic of the face. For exterior faces one has to modify the amplitude in the usual way by adapting the definition of the Euler characteristic by counting by  $1/2$  edges with one

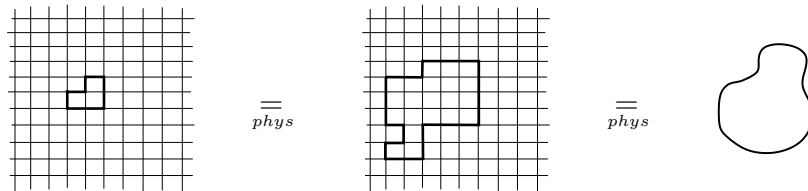


endpoint on the boundary and 0 for edges with two endpoints on the boundary and external faces.

The model could be generalized to the case of arbitrary cellular decompositions, allowing arbitrary spin network to spin network transition amplitudes and the construction of a generalized projection operator  $P$  to the physical Hilbert space. In the continuum limit, e.g. defined as in [31], the physical Hilbert space defined by  $P$  is much larger than that of  $BF$ -theory. It would be nice to show its precise relation to the Hilbert space of spin network states modulo piecewise linear homeomorphisms—a combinatorial generalization of the Hilbert space that would correspond to the quantization of the Husain–Kuchař model. We emphasize that despite of the simplicity of the model, it corresponds to a theory with infinitely many degrees of freedom. This shows that our background independence requirements are weaker than topological invariance.

In the argument at the beginning of this section we used the existence of a foliation to show the equivalence of the constraint  $BF$ -theory and the action of the Husain–Kuchař model. Is the model dependent of this foliation? The answer is no. At the classical level, the canonical analysis of the Husain–Kuchař model uses also a foliation of the manifold but then it is shown that the equations of motion imply the independence of such auxiliary structure. In our case our imposition of the constraints on  $BF$ -theory selects a direction  $u$  at each 4-simplex but this direction is not rigidly specified. As a consequence in the spin foam representation one is summing over all possible  $u$ ’s compatible with the boundary spin networks.

Notice also that the fluctuating character of  $u$  in the state-sum is imposing diffeomorphism invariance on the boundary. If we compute, e.g., the transition amplitude between the vacuum (no-spin-network state) and two states in the same (piecewise linear homeomorphism) equivalence class the contributing configurations of  $u$  will be different. Perhaps the simplest example is already illustrated by Figure 6. If we think of a single 4-simplex as our spacetime then different choices of  $u$  correspond to all the different ways of drawing the  $\theta$ -spin network on the boundary of a single 4-simplex.



**Figure 7.** Diagrammatic representation of two distinct Wilson loops that will be physically equivalent in our toy model. In general any two spin network states differing by a piecewise linear homeomorphism will be physically equivalent, i.e, their difference will be in the kernel of  $P$ . In the continuum limit one can extend the equivalence to smooth graphs as the one shown on the right.

Amplitudes in this model are crossing symmetric in the sense of [32, 33], i.e. we are imposing 3-diffeomorphism invariance in any arbitrary slicing. The purpose of this simple example is to show how the requirement of 3-diffeomorphism invariance is directly related to the values of the face and vertex amplitudes in the special configurations studied in the previous subsection. In other words, the non triviality of gravity transition amplitudes (or more precisely its generalized projection operator  $P$ ) should be encoded in the details of the vertex amplitude for the configurations that have been avoided *de facto* in this model and otherwise agree with it. From the point of view of the first part of this paper, the model does not pose any difficulties since only Lagrange multipliers are constrained to be zero directly.<sup>†</sup> There is thus no additional factor from the constraint algebra and the lower dimensional amplitudes agree with those of  $BF$ -theory. This is in agreement with the minimal requirements used here, which thus turn out to be sufficient in this case.

## 6. Discussion

The aim of this paper is to point out the need to check the anomaly problem for spin foam models which in this context means that lower dimensional simplex amplitudes are not free but essentially fixed. This is illustrated by the toy model of Section 2 which demonstrates that without the correct amplitudes non-physical results arise.

In order to find an anomaly-free formulation one has to study the constraint algebra which in spin foam models for gravity, formulated as a restricted  $BF$ -theory, generally involves second class constraints. For a complicated constraint algebra this requires a non-trivial function in the measure which has been overlooked before. We have proposed a way to address the issue of computing the measure in a simplicial theory using the classical phase space structure. To get the face and edge amplitude of the discrete spin foam model we have seen that it is helpful to introduce polar coordinates for the  $B$ -field since it provides a direct link to the spin parameters of the state sum. Explicit calculations require several integrations which, if not possible to be done explicitly, are well-suited for a stationary phase approximation. A detailed study is necessary in order to be able to judge if amplitudes of a particular model would be finite.

Even though we have not explicitly computed the anomaly free measure for spin foam models of gravity we have pointed out that some minimal requirements from the related condition of background independence can be imposed that severely restrict the value of the face and edge amplitudes. The restrictions imposed by background

<sup>†</sup> In the canonical treatment of  $BF$ -theory,  $B_{a0}^i$  are Lagrange multipliers of the constraints  $F_{ab}^i \approx 0$ . The additional second class constraints for the Husain–Kuchař model (vanishing time-components of  $B$  and their momenta) commute with the original first class constraints of  $BF$ -theory and have constant bracket among themselves.

independence in the way presented here seem to imply that bubble divergences in spin foam models for gravity are linked with the gauge action of diffeomorphisms, and rule out some proposals in the literature. In particular, without modification of the singular edge amplitude (associated with edges bounded by less than four faces), the finite normalizations for both the Riemannian and Lorentzian Barrett–Crane model proposed in [20, 34] are to be regarded as formulations where the diffeomorphism gauge symmetry has been broken by an anomalous path integral measure.<sup>†</sup> Other anomalous formulations are: the new normalization of the Barrett–Crane model proposed in [18] to improve the convergence properties of the previous model and Model A in [35]. This seems to severely limit the physical relevance of such proposals. Model B in [35] is the only normalization of the Barrett–Crane model which satisfies the minimal requirements of background independence presented here and is the one naturally arising in the quantization of Plebanski’s *Spin*(4) formulation of gravity presented in [4]. However, a non trivial modification of the edge amplitude for generic edges should appear due to the contribution of the simplicity constraints as argued in Section 3.

The value of the correct edge amplitude can be determined if we understand the canonical algebra of simplicity constraints so that the appropriate determinant as in (8) is included. This is a complicated issue as it might require the understanding of the canonical formulation of the simplicial model. Perhaps the ideas of Gambini and Pullin in the context of their *consistent discretization* formulation might shed some new light on this issue. Unfortunately the canonical formulation of discrete theories seems rather complicated in the case of Plebanski’s formulation at this stage. An interesting alternative procedure to deal with constrained systems is the projection-operator approach [36, 37, 38]. Since this method allows to deal with first and second class constraints on an equal footing, it may be possible to sidestep some of the difficulties mentioned above.

The well understood results in 3-dimensional gravity [7] and those of Section 5.2.1 suggest that the discretization does not completely fix the diffeomorphism gauge transformations as it is usually assumed. In the case of the Barrett–Crane model (any other model could be analyzed in this way) we have shown that the minimal requirements for background independence already imply the presence of certain bubble divergences that have no physical content. For instance, even in our toy (Husain–Kuchař) spin foam model divergences can not be avoided without some extra manipulation.

<sup>†</sup> This (anomalous) finite normalization of the Barrett–Crane model was naturally obtained in the context of the group field theory (GFT) formulation of the model. One could modify the amplitudes of singular edges and make the model anomaly free in the sense described here. In this way there will be divergences but of a rather simple kind (only isolated vacuum bubble will diverge and could be easily renormalized). However, the naturality argument in relation to a GFT formulation would not stand in this case.

These divergences will be present in any spin foam model for a background independent theory and the way to deal with them is by appropriate gauge fixing conditions. For this, one needs to understand in a precise manner the action of diffeomorphisms in the context of the simplicial models. Our minimal requirements of background independence of Section 5.2.1 are closely related to the action of 3-diffeomorphisms in loop quantum gravity (it is tempting to think that due to the fact that our requirements hold for any ‘slicing’ of the spin foam one is imposing a ‘bit’ of 4-diffeos in this picture). In this sense, what is left to understand is the old questions: where is the remnant gauge transformation encoded in the action of the Hamiltonian constraint in the canonical framework?, and can we expect to be able to separate the physical dynamics from the gauge evolution by a closer analysis of the vertex amplitude?

In the Barrett–Crane model we constrain the  $B$  field to be given by a simple bivector field derived from a tetrad. At the spin foam level,  $Spin(4)$  representations are constrained to simple representations  $j \otimes j^*$ . The essential questions are: what part of ‘translational’ gauge freedom (20) in the  $B$ ’s of  $Spin(4)$   $BF$ -theory remains after the implementation of the simplicity constraints? Namely: Is this gauge symmetry remnant fully encoded in the equivalence class of spin foams defined by Baez?, or is there also a symmetry that can change the values of the representation labels as in  $BF$ -theory?

In the first scenario the gauge divergence structure of the model does not seem problematic. We have seen in Section 3 that in addition to the Faddeev–Popov factor coming from gauge fixing first class constraints the measure should be modified when implementing the simplicity constraints. From the analysis of Section 5.2.1 we conclude that this modification should involve the value of the edge amplitude for non-singular edges. The implementation of the simplicity constraints could modify the amplitudes in a way that would make non-singular bubble amplitudes finite. One could think of the non trivial damping edge amplitude of the finite model [10] as arising in this way for non-singular edges. These factors must not arise for singular edges bounding only two non trivial faces as our consistency argument fixes the edge amplitude to unity in these cases. In this scenario the divergence structure of the (non-gauge fixed) amplitudes would be similar to that of our toy model of Section 5.2.1: only the vacuum bubbles as the ones represented in Figures 1 or 3 would be the divergent contributions to the amplitudes and can be easily regularized.

In the second scenario, it is appealing to think that the sum over representations ‘flowing’ inside a bubble is un-physical and its contribution corresponds to the diffeomorphism gauge volume. Gauge fixing will correspond to dropping these redundant sums and replace them by the appropriate Faddeev–Popov determinants of Section 3. In the case of  $BF$ -theory topological invariance follows from the triviality of these Faddeev–Popov determinants. In the case of gravity these factors should depend in a non-trivial fashion on the spins as the correct model must contain local excitations.

## Acknowledgments

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## Appendix: A degenerate sector of gravity

In this appendix we illustrate how the study of the integrals that define the amplitude of spin foam models obtained by discretizing a continuous action can be used to find the correct normalization of the spin foam measure (in the large spin limit). The example presented here is a little bit more involved than the one of Subsection 5.1.1 and is closely related to the Barrett-Crane model. Even though the analog of simplicity constraints is present in this example, it should be pointed out that our analysis here has an important limitation: we are not including the determinant factors appearing in (8) produced by the implementation of second class constraints in the path integral. This example is however meant as a computation that illustrates the difficulties involved in the analysis. We have kept this calculation as an appendix hoping that the technique can be useful for further developments.

A degenerate sector of Euclidean gravity can be obtained by introducing the additional constraint [40]  $B_i^L = V_i^j B_j^R$  into  $SO(4)$   $BF$ -theory where  $B^L$  and  $B^R$  are  $su(2)$  valued components of the  $so(4)$  valued  $B$  according to the decomposition  $SO(4) \cong SU(2) \times SU(2)$ , and  $V$  is an  $SO(3)$ -matrix. The new action then is

$$S = \int \left( B_i^L \wedge F_i^L + B_i^R \wedge F_i^R + \lambda^i \wedge (B_i^L - V_i^j B_j^R) \right)$$

where  $F_i^L$  and  $F_i^R$  denote the curvatures of the left and right component of the connection, respectively. In this form,  $\lambda_{ab}^i$  and  $V_i^j$  appear as new Lagrange multipliers in addition to the multipliers  $B_{0a}^{L/R,i}$  and  $A_0^{L/R,i}$  of the original  $BF$ -theory. Only the multipliers  $B_{0a}^{L,i}$  and some components of  $\lambda_{ab}^i$  are restricted by the new constraints and in order to obtain a constrained system on a symplectic phase space we have to add their momenta to the canonical variables together with constraints requiring them to be zero. For a discussion of the finiteness of the resulting model it is most interesting to see whether or not additional factors in the measure depend on components of  $B$  with a non-trivial scaling behavior since this would affect the large- $j$  behavior of face amplitudes. This has to be expected here because varying  $V_i^j$  yields a constraint which restricts components of the multipliers  $\lambda_{ab}^i$  and is linear in  $B$ . Since we had to add

momenta of these components of  $\lambda_{ab}^i$ , the constraint algebra will contribute positive powers of  $B$  to the measure which would enhance a divergence. Here, however, we do not discuss the constraints in detail but rather ignore the additional factor and derive the large- $j$  behavior of the face amplitude for the naive spin foam quantization with trivial multiplier measure. While the resulting amplitude would not be correct, our aim here is solely to compare it with the result of [4] where this factor has been ignored, too.

To include the additional constraint into a spin foam quantization we impose it face-wise such that any face  $f$  carries a matrix  $V_f$ . This gives the state sum

$$\begin{aligned} Z &= \int \prod_e d^3 g_e^L d^3 g_e^R \prod_f d^3 B_f^R d^3 V_f \exp \left( i \operatorname{tr} (B_f^R (U_f^R + U_f^L V_f)) \right) \\ &= \int \prod_e d^3 g_e^L d^3 g_e^R \prod_f d^3 B_f^L d^3 B_f^R d^3 V_f d^3 \lambda_f \exp \left( i \operatorname{tr} (B_f^L U_f^L + B_f^R U_f^R + \lambda_f (B_f^L - V_f B_f^R)) \right) \end{aligned}$$

where  $\lambda_f$  are Lagrange multipliers and the edge holonomies  $(g_e^L, g_e^R)$  form the holonomies  $U_f^L$  and  $U_f^R$  along closed loops. Integrating over  $\lambda_f$  yields  $\delta$ -functions which will be solved after introducing polar coordinates  $(r^{L/R}, \vartheta^{L/R}, \varphi^{L/R})$  for  $B^{L/R}$  and Euler angles  $(\psi, \theta, \phi)$  for  $V$ . The  $\delta$ -functions then imply  $r_f^L = r_f^R$  for all faces  $f$ , and  $\vartheta^L$  and  $\varphi^L$  will be given as functions  $\vartheta^L = F(\vartheta^R, \psi - \varphi^R, \theta)$  according to  $\cos \vartheta^L = \sin \theta \sin \vartheta^R \sin(\psi - \varphi^R) + \cos \theta \cos \vartheta^R$  and  $\varphi^L = G(\vartheta^R, \varphi^R, \psi, \theta, \phi)$ . Choosing again a gauge for  $U^{L/R}$  without loss of generality, we obtain

$$\begin{aligned} Z &= \int \prod_e d^3 g_e^L d^3 g_e^R \prod_f d^3 B_f^L d^3 B_f^R d^3 V_f \delta^3(B_f^L - V_f B_f^R) \exp \left( i \operatorname{tr} (B_f^R U_f^R + B_f^L U_f^L) \right) \\ &= \int \prod_e d^3 g_e^L d^3 g_e^R \prod_f dr_f^L d\vartheta_f^L d\varphi_f^L (r_f^L)^2 \sin \vartheta_f^L dr_f^R d\vartheta_f^R d\varphi_f^R (r_f^R)^2 \sin \vartheta_f^R d\psi_f d\theta_f d\phi_f \sin \theta_f \\ &\quad \times ((r_f^L)^2 \sin \vartheta_f^L)^{-1} \delta(r_f^L - r_f^R) \delta(\vartheta_f^L - F(\vartheta_f^R, \psi_f - \varphi_f^R, \theta_f)) \delta(\varphi_f^L - G(\vartheta_f^R, \varphi_f^R, \psi_f, \theta_f, \phi_f)) \\ &\quad \times \exp \left( -i(r_f^L \cos \vartheta_f^L \sin \frac{c_f^L}{2} + r_f^R \cos \vartheta_f^R \sin \frac{c_f^R}{2}) \right) \\ &= \int \prod_e d^3 g_e^L d^3 g_e^R \prod_f dr_f^R d\vartheta_f^R d\varphi_f^R (r_f^R)^2 \sin \vartheta_f^R d\psi_f d\theta_f d\phi_f \sin \theta_f \\ &\quad \times \exp \left( -i r_f^R (\cos F(\vartheta_f^R, \psi_f - \varphi_f^R, \theta_f) \sin \frac{c_f^L}{2} + \cos \vartheta_f^R \sin \frac{c_f^R}{2}) \right). \end{aligned}$$

We now have to perform the angle integrations, which is effectively a four-dimensional integral since the  $\phi$ - and  $\psi + \varphi^R$ -integrations are trivial, in order to find the face amplitude after discretizing  $r^R$ . One can use the saddle point approximation, but has to be careful because the integrand has zeroes due to the sines. Disregarding the zeroes, one would expect the asymptotic behavior of the angle integral to be  $(r^R)^{-2}$  because every one-dimensional integration contributes  $r^{-\frac{1}{2}}$  in the standard stationary phase result. This factor would cancel the  $(r^R)^2$  from the measure and thus yield 1 as the large- $j$  behavior of the face amplitude. In [4] a face amplitude  $A_f(j) = 1$  has been obtained for

this model by different means, however for a different discretization of the action which is not accessible to our methods. Nevertheless, there is agreement provided that the modified discretization in [4] leads to the standard stationary phase result. An analysis similar to that of Section 5.2.1 would lead to the conclusion that such a face amplitude is in conflict with background independence. This should not be surprising as there is no reason for amplitudes to be anomaly free when we do not include the correct measure in the presence of second class constraints. At present we do not see whether doing so in this case would lead to amplitudes in agreement with background independence.

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